ON THE IMMERSION OF DIGRAPHS IN CUBES

BY

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ABSTRACT

We consider some problems concerning generalizations of embeddings of acyclic digraphs into n-dimensional dicubes. In particular, we define an injection i from a digraph D into the n-dimensional dicube Q_n to be an *immersion* if for any two elements a and b in D there is a directed path in D from a to b iff there is a directed path in Q_n from $i(a)$ to $i(b)$. We further define the immersion to be *strong* iff there is a way of choosing these paths so that paths in Q_n corresponding to arcs in D have disjoint interiors, and we introduce a natural notion of "minimalit/" on the set of arcs of a digraph in terms of its paths. Our main theorem then becomes: *Every (minimal) n-element acyclic digraph can be (strongly) immersed in* Q_n. We also present examples of n-element digraphs which cannot be immersed in Q_{n-1} and examples of $9n$-element non-minimal digraphs which cannot be strongly immersed in Q_{10n-1}. We conclude with some applications.

1. Introduction

In this paper we shall consider some problems in connection with the embedding of directed graphs (digraphs) into n-dimensional directed cubes (dicubes). We shall be especially interested in full embeddings and certain related embeddings.

We begin by noting that all n-dimensional dicubes are acyclic so we shall hereafter restrict ourselves to finite acyclic digraphs. We shall also assume, for technical reasons, that between any two distinct points there is at most one arc and that there is never an arc from a point to itself. Because of this, we may represent a digraph D by the pair $(V(D), A(D))$ where $V(D)$ (the vertices of D) is any finite non-empty set, and $A(D)$ (the arcs) is any appropriate subset of $V(D) \times V(D)$. As usual, we define a point $v \in V(D)$ to be a **global source (target)** if for all $w \neq v$ we have $(v, w) \in A(D)$ $((w, v) \in A(D))$. We define a sequence $p = (v_0, v_1, \cdots, v_n)$ of elements of $V(D)$ to be a **path of length** n iff $i < n \rightarrow (v_i, v_{i+1}) \in A(D)$, and we denote the set

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of paths in D by $P(D)$. We define v_0 and v_n to be the **left** and **right endpoints** of the above path; the remaining points constitute the path's **interior**. Two paths are **essentially disjoint** iff their interiors are disjoint. The **transitive closure** D^* of a digraph D is the digraph $(V(D), A^*(D))$ where $(v, w) \in A^*(D)$ iff v and w are the left and right endpoints respectively of some path $p \in P(D)$.

Now, given two digraphs G and H, we say that an injection $\phi: V(G) \to V(H)$ is an **immersion** of G into H iff it is a full embedding of G^* into H^*. That is, ϕ is an immersion if for any $u, v \in V(G)$ there is a path from u to v in $P(G)$ iff there is a path from $\phi(u)$ to $\phi(v)$ in $P(H)$. We note that an immersion ϕ of G into H need not be a homomorphism of G into H, i.e., $(v, w) \in A(G)$ does not imply $(\phi(v), \phi(w)) \in A(H)$.

It is also of interest to require that paths in H corresponding to the arcs in G be chosen to be essentially disjoint. Thus we define a pair $\Phi = (\phi, \phi^+)$ to be a **strong immersion** of G into H iff ϕ is an immersion of G into H and ϕ^+ is a function from $A(G)$ into $P(H)$ such that $(u, v) \in A(G)$ implies that $\phi(u)$ and $\phi(v)$ are the left and right endpoints of $\phi^+(u, v)$ and such that the images of any two distinct members of $A(G)$ are essentially disjoint.

As we have mentioned, we shall be interested in immersing various digraphs into n-dimensional dicubes, so it is convenient to deal with a particular representation of this dicube. We denote the n-dimensional dicube by Q_n, we let $V(Q_n)$ be the set of functions from $(0, 1, \cdots, n-1)$ into $\{0, 1\}$, and we let $A(Q_n)$ be the set of pairs (f, g) of these functions such that f and g agree at all but one member of their domain and on that point f takes the value 0 and g the value 1. If $s = (f_1, f_2, \cdots, f_n)$ is any sequence of points in Q_n, then the **canonical matrix** of s is the $m \times n$ matrix in which $a_{ij} = f_i(j)$. Note that for distinct $f, g \in V(Q_n)$ we have $(f, g) \in A^*(Q_n)$ iff $f(j) = 1$ implies $g(j) = 1$ for every j.

Finally, we remember that we did not allow duplication of arcs between the same two points. Since we are interested in transitive closures, it is not unreasonable to also eliminate arcs between points which are already joined by longer paths. Thus we define a digraph G to be **minimal** (in the sense that it is the "smallest" graph which generates its particular transitive closure) iff for no arc $(u, v) \in A(G)$ does there exist a path $p \in P(G)$ of length greater than 1 from u to v.

2. Main theorem

In this section we show that every n-element acyclic digraph can be immersed in an n-dimensional dicube and that if the digraph is also minimal, the immersion can be extended to a strong immersion. Later, we shall show that there exist $(n + 1)$-element digraphs which cannot be immersed in Q_n and there exist (non-minimal) n-element graphs which cannot be strongly immersed in Q_n.

THEOREM 2.1. *If D is any (minimal) n-element acyclic digraph, then:*

(a) *D can be (strongly) immersed in Q_n;*

(b) *if D^* has either a global source or a global target, then D can be (strongly) immersed in Q_{n-1};*

(c) *D can be strongly immersed in Q_{2n}.*

PROOF. Since D is acyclic, we can enumerate $V(D)$ as $v_0, v_1, \cdots, v_{n-1}$ in such a way that $i < j$ implies $(v_i, v_j) \notin A^*(D)$. Then for each $i < n$ we let $\phi(v_i)$ be the function $f_i \in Q_n^*$ defined by

$$f_i(m) = \begin{cases} 1 \text{ if } (v_m, v_i) \in A^*(D) \text{ or } m = i \\ 0 \text{ otherwise.} \end{cases}$$

We note that $f_i(m) = 0$ for $m < i$.

We first prove that ϕ is an immersion. Suppose $(v_i, v_j) \in A(D)$, so $i > j$. If $f_i(m) = 1$, then either $(v_m, v_i) \in A^*(D)$ or $m = i$. If $(v_m, v_i) \in A^*(D)$, then $(v_m, v_j) \in A^*(D)$. On the other hand, if $m = i$, then $(v_m, v_j) \in A(D) \subset A^*(D)$. Thus, in either case, $f_j(m) = 1$. Hence, $(f_i, f_j) \in A^*(Q_n)$. Of course, it follows by transitivity that if $(v_i, v_j) \in A^*(D)$, then $(f_i, f_j) \in A^*(Q_n)$. Conversely, suppose $(f_i, f_j) \in A^*(Q_n)$. Then $f_j(i) = 1$ since $f_i(i) = 1$. Therefore $(v_i, v_j) \in A^*(D)$.

Next assume that D is minimal. We must describe a mapping ϕ^+ from $A(D)$ into $P(Q_n)$ which yields paths with nonintersecting interiors. It will, of course, be sufficient to choose these paths in such a way as to insure that given any interior point of any path it is possible to uniquely determine both endpoints of the path. For simplicity we shall abbreviate "$\phi^+((v_i, v_j))$" by "$\phi^+(i,j)$".

We begin by noting that the canonical matrix of the image $(f_0, f_1, \cdots, f_{n-1})$ of ϕ is triangular. That is, it has only ones on its diagonal and only zeros below the diagonal. Then for any arc $(v_s, v_t) \in A(D)$ we choose the first interior point of the path $\phi^+(s,t)$ from f_s to f_t to be the function $g_{st}^1 \in V(Q_n)$ defined by

$$g_{st}^1(m) = \begin{cases} 1 & m = t \\ f_s(m) & \text{otherwise.} \end{cases}$$

Now given any interior point g of any path $\phi^+(i,j)$, we note that j will always be be the least number such that $g(j) = 1$. To determine i from g will be more difficult, and it is for this reason that we must choose the remaining interior points of $\phi^+(s,t)$ carefully. We first look at *all* points v_i which have arcs to v_t, and we define

$$C_t = \{i : (v_i, v_t) \in A(D)\}$$

to be the **critical set** for t. It then follows immediately from the definition of ϕ and the minimality of D that

$$i,j \in C_t \to [(f_t(i) = f_t(j) = 1) \wedge (f_i(j) = 1 \leftrightarrow i = j)].$$

Now let $g^c_{st} \in V(Q_n)$ be the function defined by

$$g^c_{st}(m) = \begin{cases} f_t(m) & m = s \vee m \notin C_t \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $i \in C_t$ we have $g^c_{st}(i) = 1$ iff $i = s$. We now choose any path in $P(Q_n)$ from g^1_{st} to g^c_{st} and let that be the initial part of the interior of $\phi^+(s,t)$. It follows immediately that such a path always exists and that given any function $g \in V(Q_n)$ in this part of a path $\phi^+(i,j)$ it is easy to first determine j and then, by looking at C_j, to find i.

Finally, to choose the remainder of the path $\phi^+(s,t)$, we note that the submatrix of the canonical matrix consisting of those a_{ij} such that $i,j \in C_t$ is simply an identity matrix, and that completing paths of the form $\phi^+(i,t)$ is simply a matter of filling in the rows one step at a time. Of course this must be done in such a way as to continue to allow us to recover the left endpoint of the path, but this is now no great problem. One easy way is to first fill in from left to right starting at the diagonal point (which is already 1) until the row is completed and then to fill in the remaining points, again from left to right, starting from the left endpoint. The details needed to formally describe this procedure are tedious and are left to the reader. We only note that given any point f of this kind on some path $\phi^+(i,j)$ we first find j as usual. We then look at $C_j = \{c_0, c_i, \cdots c_m\}$ where the c_k are arranged in increasing order, and it is easily seen that if there exist a $k \leqq m$ such that $f(c_{k-1}) = 0$ and $f(c_k) = 1$, then i must be c_k, otherwise i must be c_0. We note also that we have made strong use of the fact that D is minimal by assuming that the submatrix associated with the critical set is always the identity matrix.

Now suppose that D^* has a global source v_{n-1}. Then for each $i < n$ we have $f_i(n-1) = 1$ so we may restrict each f_i to the $n-1$ coordinates $\{0, 1, \cdots, n-2\}$ to obtain an immersion into Q_{n-1}. If the original immersion is strong, then the restriction to $n-1$ coordinates does not affect the paths, and the immersion remains strong. If, on the other hand, D^* has a global target, then we may first "reverse" the arcs of D, then immerse the reversed graph (strongly) into Q_{n-1}, and, finally, reverse the arcs in Q_{n-1} to obtain the desired immersion.

We conclude the proof of Theorem 2.1 by constructing a strong immersion (θ, θ^+) of D into Q_{2n}. As before, we enumerate the vertices $v_0, v_1, \cdots, v_{n-1}$ of D in any way such that $1 < j \to (v_i, v_j) \notin A^*(D)$, and for each $i < n$ we let $\theta(v_i)$ be the point $f_i \in V(Q_{2n})$ defined by

$$f_i(j) = \begin{cases} 1 & i = j \vee (v_j, v_i) \in A^*(D) \vee j \geqq n + i \\ 0 & \text{otherwise.} \end{cases}$$

Then for any pair $(v_s, v_t) \in A(D)$ we define the path $\theta^+((v_s, v_t))$ as follows. As before, let the first interior point of the path be the point $g_{st} \in V(Q_{2n})$ defined by

$$g_{st}(i) = \begin{cases} 1 & i = t \\ f_s(i) & \text{otherwise.} \end{cases}$$

Now, however, it is sufficient to obtain the remaining interior points of the path b filling in from left to right the appropriate ones in the row corresponding to f_s in the canonical matrix corresponding to the image of θ. Then given any interior point g of a path $\theta^+((v_s, v_t))$, it is easily seen that t is the least number such that $g(t) = 1$ and s is the least number such that $i \geqq n + s \to s(i) = 1$. ∎

3. Counterexamples

Clearly we do not need $2n$ dimensions to strongly immerse an n-element digraph. If we look at the paths we used in the case of minimal graphs, we see that all we really need is some mechanism to determine the origins of these paths, and thus, for example, we need not worry about elements which are local targets. However, in other aspects this theorem is best possible. To show this we first prove a lemma which will allow us to extend certain small counterexamples to higher dimensions.

LEMMA 3.1. *Suppose E is an $(n+2)$-element digraph, D is the full subgraph over some n-element subset of $V(E)$, and the points p and q in $V(E) - V(D)$ satisfy:*

(a) $(p, q), (q, p) \notin A^*(E)$, *and*

(b) $r \in V(D) \to (p, r), (q, r) \in A^*(E)$.

Then E can be (strongly) immersed in Q_{2n+2} only if D can be (strongly) immersed in Q_{2n}.

Proof Suppose the function ϕ (pair (ϕ, ϕ^+) strongly) immerses E in Q_{2n+2}. If, as usual, we denote $\phi(p)$ by f_p, then we see that since p and q are incomparable in

E^*, we may assume without loss of generality that $f_p(n) = f_q(n+1) = 1$ and $f_p(n+1) = f_q(n) = 0$. But this implies by condition (b) above that for each $r \in V(D)$ we have $f_r(n) = f_r(n+1) = 1$. This, in turn, implies that for any interior point f along a path between the images under ϕ of any two points in D we also have $f(n) = f(n+1) = 1$. Thus if ϕ is a (strong) immersion of E into Q_{n+2}, then the function ϕ^* defined by restricting f_p to the subset $\{0, \cdots, n-1\}$, i.e.,

$$\phi^*(p) = f_p \,|\, \{0, \cdots, n-1\}$$

is an (can be extended to a strong) immersion of D into Q_n. ∎

We note the fact that this lemma is also true if we have three or four incomparable elements which satisfy (b) instead of only two. This is true essentially because to fully embed two, three, or four incomparable elements requires two, three, or four dimensions respectively. On the other hand up to six incomparable points points can be embedded into four dimensions, so if we used six points in the hypothesis and assumed we had an immersion of E into Q_{n+6}, then the best we could obtain would be an immersion of D into $Q_{n+6-4} = Q_{n+2}$. For our purposes it will be sufficient to look at the cases where only two points are added.

While it is clear that digraphs with paths of length N cannot be immersed in Q_{n-1}, we show that even in the more general case n dimensions may be needed.

THEOREM 3.2. *For each $n > 1$ there exists an n-element acyclic digraph D_n which contains only paths of length less than $n/2$ but which cannot be immersed in Q_{n-1}.*

PROOF. For each $n > 0$ set

$$V(D_{2n}) = \{v_i^m : m < n \wedge i < 2\} \text{ and } V(D_{2n+1}) = V(D_{2n}) \cup \{v_2^{n-1}\},$$

and in both cases set $A(D_n) = \{(v_i^r, v_j^s) : r + 1 = s\}$. It now follows by induction using (3.1), where the cases $n = 2, 3$ are checked by inspection, that no D_n can be immersed in Q_{n-1}. (See Fig. 1 for examples.) ∎

We next show that not every n-element digraph can be strongly immersed in Q_n and that not every n-element digraph whose transitive closure contains a global source can be strongly immersed in Q_{n-1}. All of our counterexamples will be based on one graph which we present next.

LEMMA 3.3. *The digraph D in Fig. 2 cannot be strongly immersed in Q_4.*

PROOF. Suppose there is a pair $(\phi : D \to Q_4, \phi^+)$ which is a strong immersion. We define the **level** of a point f in Q_n to be the number of points on its domain on

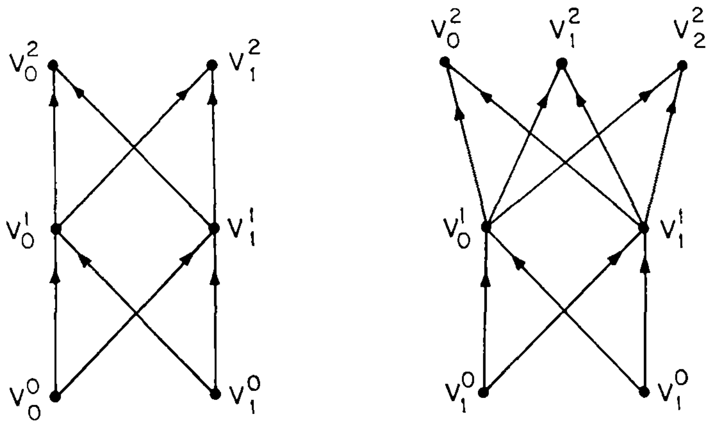

Fig. 1. The graphs D_6 and D_7

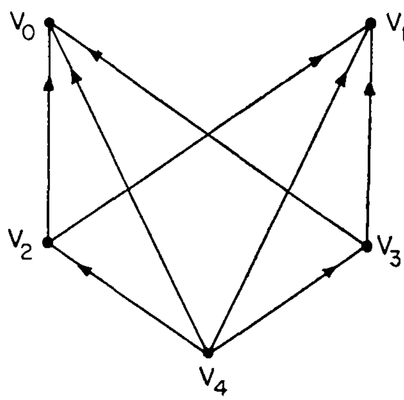

Fig. 2. The digraph D

which it takes the value 1 and, as earlier, we use f_i to denote $\phi(v_i)$. Because f_0 and f_1 are incomparable and are each the target of three distinct points, they must have level exactly 3. Thus, without loss of generality, we may represent them by the sequences 1110 and 1101 respectively. Now suppose f_2 has level 2. Then the only possible point in Q_4 that it can be is the one represented by the sequence 1100. This then implies that f_3 must have level 1. But the only points of level 1 which are below both f_0 and f_1 are also below f_2, so there is no consistent way to choose f_3. Thus we may assume that f_2 and f_3 both have level 1, and, of course, that f_4 has level 0.

However, since paths proceed one level at a time, it is easily seen that there must exist in the range of ϕ^+ exactly six disjoint paths through level 2 which terminate at either f_0 or f_1.On the other hand, there exist only six points of level 2, and it is again easily seen that one of these does not connect to either f_0 or f_1. Thus the six paths must pass through five points and therefore cannot be essentially disjoint. ■

But this gives us:

THEOREM 3.4. *For every $n \geqq 5$ there exists an n-element acyclic digraph with a global source which cannot be strongly immersed in Q_{n-1}.*

PROOF. To obtain an appropriate 6–element digraph add a new point below v_4 in D above. Then either add more such points or use (3.1) to obtain the desired larger digraphs. ■

To deal with n-element digraphs and Q_n we need:

LEMMA 3.5. *The digraph D^+ in Fig. 3 cannot be strongly immersed in Q_9.*

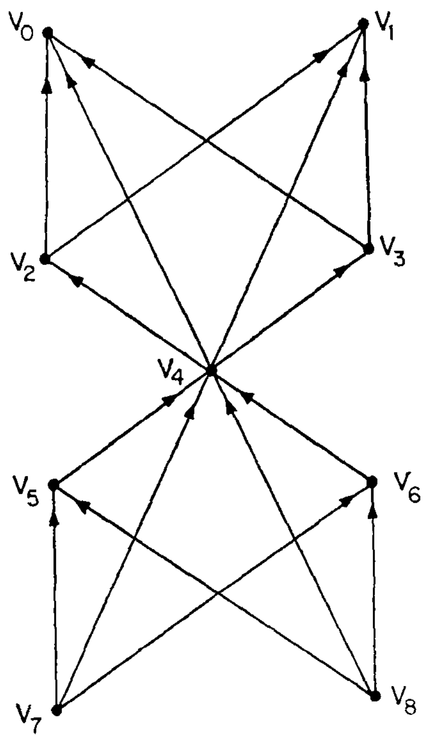

Fig. 3. The digraph D^+

PROOF. Suppose there is some pair (ϕ, ϕ^+) which is a strong immersion. Consider the level of $\phi(v_4)$. If it is at least 5, then we may delete 5 of the coordinates on which it is 1, and the pair (ϕ, ϕ^+) restricted to the remaining coordinates will be a strong immersion of a copy of D (the graph constructed in (3.3)) into Q_4 which by (3.3), is impossible. On the other hand, if the level of $\phi(v_4)$ is less than 5, we may delete 5 of the coordinates on which it is 0 and, by again restricting (ϕ, ϕ^+) to the remaining set of coordinates, obtain a strong immersion of the reverse of D into Q_4. However, this, by symmetry, is also impossible. ∎

Using this, we have:

THEOREM 3.6. *For every $n \geqq 9$ there exists an n-element acyclic digraph which cannot be strongly immersed into Q_n.*

PROOF. To obtain an appropriate ten element graph it is sufficient to add to D^+ above a new point which is below v_4 and above v_5, v_6, v_7, and v_8. For larger digraphs again we may either add more such points or use (3.1). ∎

Finally, we show that even Q_{n+1} may not be enough.

THEOREM 3.7. *For every $n \geqq 1$ there exists a 9n-element acyclic digraph which cannot be strongly immersed in any dicube of dimension less than* $10n$.

PROOF. The digraphs in question will be built up by "stacking" n copies of D^+ one above the other. Thus for each $0 \leqq m < n$ we let $D_m^+ = (V_m^+, A_m^+)$ be a copy of D^+ where we let $V_m^+ = \{v_0^m, v_1^m, \cdots, v_8^m\}$, and we let A_m^+ be as in Figure 3. Now define $D_n^* = (V_n^*, A_n^*)$ by setting $V_n^* = \bigcup_{m<n} V_m^+$ and

$$A_n^* = \bigcup_{m<n} A_m^+ \cup \{(v_i^k, v_j^{k+1}) : i \in \{0,1\} \text{ and } j \in \{7,8\}\}.$$

The proof that a dicube of at least $10n$ dimensions is required to strongly immerse this graph is inductive, and the proof of the inductive stage is similar to that of (3.5). We note immediately that the case $n = 1$ is simply (3.6). Suppose now the theorem is true for $n = m$, and strongly immerse D_{m+1} in some dicube using some function ϕ. Then by our induction hypothesis there will be at least $10m$ coordinates on which either $\phi(v_7^{m-1})$ or $\phi(v_8^{m-1})$ will take on the value 1. But this implies that there is a set of at least $10m$ coordinates on which *both* $\phi(v_0^m)$ and $\phi(v_1^m)$ take on the value 1, and we know from (3.5) that to complete the strong immersion we require at least 10 more coordinates thus giving a total $10(m + 1)$. ∎

We note that in defining D^+ and D_n^* we did not have to look at the transitive closures of the digraphs involved. It may well be that doing so might lead to stronger results. On the other hand, by not doing so, we end up with digraphs which are very close to minimal. That is, suppose we define the **complexity** of a digraph to be the largest number n for which there exists a path $(v_0, v_1, \cdots, v_{n+1})$ such that (v_0, v_{n+1}) is an arc. Then the minimal graphs are those of complexity 0 and we see immediately that all the D_n^* have complexity only 1. Thus they are very close to minimal, and yet the number of "extra" dimensions required to immerse them is arbitrarily large.

4. Applications

The problem of determining whether or not a given graph or digraph can be embedded in a complete graph or digraph is, of course, trivial. However, the corresponding problem for cubes seems to be quite difficult. If G is a graph which can be embedded in some cube, then certainly G is bipartite. However, $K_{2,3}$ shows that this necessary condition is not sufficient since it is easy to see that $K_{2,3}$ is not embeddable in any cube. Recently, however, Djoković [1] has shown that G can be embedded in a cube so that distance is preserved if and only if G is bipartite and satisfies the following condition: $G(a,b)$ is closed whenever a and b are adjacent vertices in G. Here, $G(a,b) = \{v \in V(G) : d(v,a) > d(v,b)\}$, and $W \subset V(G)$ is closed if, for all $a, b \in W$ and $v \in V(G)$

$$d(a,v) + d(v,b) = d(a,b) \to v \in W.$$

If D is a digraph, then the following two conditions are necessary for D to be embeddable in a dicube.

Condition 1. The "integral" around any undirected cycle in D is zero — that is, traversing any undirected cycle you will encounter exactly as many arcs in their proper direction as you encounter arcs in their reverse direction. (In particular, D is bipartite as an undirected graph.)

Condition 2. Let $(v,w) \in A^*(D)$ and let $p_1, \cdots, p_k \in P(D)$ be the set of all paths from v to w. Then $k \leqq \lambda!$ where λ is the length of any of the paths p_i (all of whose lengths are the same by Condition 1). These conditions are not sufficient since the digraph in Fig. 4 satisfies Conditions 1 and 2 but can't be embedded in a dicube. We do not know of any necessary and sufficient conditions for embedability of a digraph in a dicube.

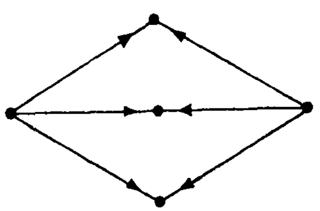

Fig. 4

If G is a graph (D is a digraph), we call the graph G' (the digraph D') a **subdivision** if it is obtained by replacing certain edges e (arcs a) by paths (directed paths) all of whose interior points are new and which join the points originally joined by e (or a).

It is somewhat artificial to say that the triangle in Fig. 5 does not embed in a

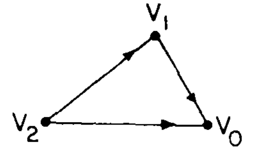

Fig. 5

cube since the triangle has a subdivision which embeds (identically!) in Q_2. Our main theorem shows that this is the case for *any* digraph.

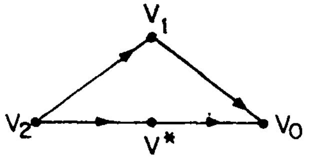

Fig. 6.

THEOREM 4.1. *Let D be an acyclic digraph with n vertices. Then D has a subdivision D' which can be embedded in Q_{2n}. If D is minimal, then it has a subdivision which can be embedded in Q_n.* ∎

This theorem is an immediate consequence of the following lemma and Theorem 2.1.

LEMMA 4.2. *Let $(\phi, \phi^+) : D \to Q_r$ be a strong immersion. Then there is a subdivision D' of D which is embeddable in Q_r.*

PROOF. Let $a = (v, w)$ be any arc of D. Then $\phi^+(a) \in P(Q_r)$, say $\phi^+(a) = (v', v_1', \cdots, v_k', w')$, where $\phi(v) = v'$, $\phi(w) = w'$, and $k \geqq 0$. Now subdivide a, replacing it with a path $\bar{a} = (v, v_1, \cdots, v_k, w)$, where $v_1, \cdots, v_k$ are new vertices. Do the same for all arcs $a \in A(D)$ obtaining a subdivision D' of D. Set $\phi'(v) = \phi(v)$ if $v \in V(D)$ and $\phi'(v_i) = v_i'$ for any vertex v_i in the interior of $\phi^+(a)$ for some arc a. Then $\phi' : D' \to Q_r$ is an embedding. ∎

Thus we have shown that by modifying slightly any digraph, we can embed it in a cube of rather low dimension. This complements a result (in the undirected case) of Graham and Pollak [2] who showed that one could also obtain an embedding by modifying the cube, rather than the graph.

They were concerned with the following problem: Assign a k-digit binary "address" to each vertex v of a graph G in such a way that $d(v, w) = d(a(v), a(w))$, where $d(v, w)$ denotes the distance from v to w in G and $d(a(v), a(w))$ is the Hamming distance from $a(v)$ to $a(w)$, i.e., the number of coordinates at which $a(v)$ and $a(w)$ differ. Such an **addressing** corresponds to an embedding of G in Q_k. Of course, this is not possible in general: for example, K_3 cannot be addressed in this way.

Graham and Pollak get around this problem by introducing the letter d in addition to 0 and 1 in the addresses, with distance computed by counting two coordinates as distinct only when one has a 1 and the other a 0. This amounts to embedding G in a cube some of whose faces have been "squashed". Moreover, the dimension of the cube is at most $s(n-1)$ where s is the maximum distance between any two vertices of G, and an algorithm is given to obtain the cube which has heuristically always produced a cube of dimension at most $n-1$.

Our addressing scheme has one serious defect: it may not lead to the shortest path. For consider the digraph E in Fig. 7.

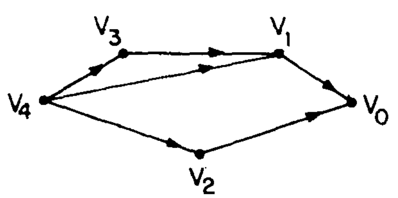

Fig. 7. The digraph E

The strong immersion θ of E into Q_{10} given by our theorem has

$$\theta(v_0) = (1,1,1,1,1,1,1,1,1,1),$$

$$\theta(v_1) = (0,1,0,1,1,0,1,1,1,1),$$

$$\theta(v_2) = (0,0,1,0,1,0,0,1,1,1),$$

$$\theta(v_3) = (0,0,0,1,1,0,0,0,1,1),$$

and

$$\theta(v_4) = (0,0,0,0,1,0,0,0,0,1).$$

(In fact, the fifth and tenth coordinates are always 1 and hence expendable, so θ is really an immersion of E into Q_8.) Presumably, we should find a shortest path from v_4 to v_0 by proceeding to that vertex whose Hamming distance from v_0 is minimal, but that would be v_3 which is just the vertex we should not go to.

Call a poset U **n-universal** iff any n-element poset P can be fully embedded in U. If we delete the condition of fullness, then obviously the n-element poset $\vec{K}_n$, whose elements $0, 1, \ldots, n-1$ are ordered in the usual way, would suffice. However, our main theorem shows that we can meet the requirement of fullness with an n-dimensional cube.

COROLLARY 4.3. *Q_n^* is an n-universal poset.*

PROOF. One of us has given a direct proof of this result elsewhere [3], using the techniques of category theory. However, by deleting loops any poset can be regarded as an acyclic digraph which is equal to its transitive closure. ∎

Note that Q_n^* is just the poset of all subsets of a set with n elements with the relation of containment.

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